

Valentine's Day maths for A Level: answers

A Level Math Revision · A Level Maths (Year 12 and Year 13) · 14 February · Teacher answers and notes

The starter is non-calculator. A calculator helps in tasks A and C.

Teacher notes

- Plot both curves on graphing software first: the reveal is part of the fun.
- In task A (d), the integral comes out negative because the curve is traced anticlockwise from $t = 0$ with x decreasing on top: discuss why we take the modulus.
- Task C works as a class experiment: shuffle five name cards ten times and compare the results with $11/30$.

Starter

1. $x^2 + y^2 = 1$

- $\cos^2 t + \sin^2 t = 1$

2. $2\cos^2 t - 1$

- Double-angle formula.

3. **1**

- $dy/dx = (dy/dt) \div (dx/dt) = 2 \div 2t = 1$ when $t = 1$.

4. **10**

- $5 \times 4 \div 2 = 10$

Task A: The cardioid

a. **(1, 0), (1, 2), (-3, 0)**

- $t = 0: (2 - 1, 0 - 0) = (1, 0)$
- $t = \pi/2: (0 - (-1), 2 - 0) = (1, 2)$
- $t = \pi: (-2 - 1, 0 - 0) = (-3, 0)$

b. **$dy/dx = (\cos t - \cos 2t)/(\sin 2t - \sin t)$; -1 at $t = \pi/2$**

- $dx/dt = -2\sin t + 2\sin 2t$ and $dy/dt = 2\cos t - 2\cos 2t$.
- At $t = \pi/2: dy/dt = 0 + 2 = 2$ and $dx/dt = -2 + 0 = -2$, so $dy/dx = -1$.

c. **Both are 0: a sharp point (cusp) at (1, 0)**

- $-2\sin 0 + 2\sin 0 = 0$ and $2\cos 0 - 2\cos 0 = 0$.
- The point stops moving for an instant, which makes the dip of the heart.

d. **$6\pi \approx 18.8$**

- $\int_0^{2\pi} (2\sin t - \sin 2t)(-2\sin t + 2\sin 2t) dt = -18.849...$
- $|-18.849...| = 18.849... = 6\pi$

Task B: A heart curve

a. **(0, 5) and (0, -17)**

- $t = 0: x = 0$ and $y = 13 - 5 - 2 - 1 = 5$.
- $t = \pi: x = 0$ and $y = -13 - 5 + 2 - 1 = -17$.

b. **(16, 4)**

- $x = 16 \times 1 = 16$, $y = 0 - 5(-1) - 2(0) - 1 = 4$.
- $\sin^3 t \leq 1$, so $x \leq 16$, with equality only at $t = \pi/2$.

c. **Replacing t by $-t$ changes the sign of x but not y**

- $\sin^3(-t) = -\sin^3 t$, and every cosine term is unchanged, so the point (x, y) goes to $(-x, y)$.

Task C: A perfect match?

a. **120**

- $5! = 120$

b. **11/30**

- $44/120 = 11/30$

c. **3/8**

- Choose the lucky friend: 5 ways; the other four all miss: 9 ways.
- $5 \times 9 = 45$, and $45/120 = 3/8$.

d. **$19/30 \approx 0.633$; $1 - 1/e \approx 0.632$**

- $1 - 11/30 = 19/30 = 0.6333\dots$
- $1 - 1/e = 0.6321\dots$: very close already for five people.

Extension

1. **16**

- $(-2\sin t + 2\sin 2t)^2 + (2\cos t - 2\cos 2t)^2 = 4 + 4 - 8(\cos t \cos 2t + \sin t \sin 2t) = 8 - 8\cos t$.
- $8 - 8\cos t = 16\sin^2(t/2)$, and $\sin(t/2) \geq 0$ for $0 \leq t \leq 2\pi$.
- $\int_0^{2\pi} 4\sin(t/2) dt = [-8\cos(t/2)]_0^{2\pi} = 8 + 8 = 16$

Every question is original and every answer was recomputed by computer before publishing.