

Fibonacci Day maths for A Level: answers

A Level Math Revision · A Level Maths (Year 12 and Year 13) · 23 November · Teacher answers and notes

The starter is non-calculator. A calculator helps in tasks A and C.

Teacher notes

- Fibonacci Day is 23 November because 11/23, written month first, reads 1, 1, 2, 3.
- In task A (c), the assumption that the limit exists matters: ask the class for a recurrence where 'solving for the limit' gives a wrong answer because the sequence diverges.
- Task C links the golden ratio to exact trigonometric values: $\cos 36^\circ = \phi/2$.
- The extension is a first look at second-order recurrences, which Further Maths students meet again.

Starter

1. **18**

- 2, 1, 3, 4, 7, 11, 18

2. **$x = (1 \pm \sqrt{5})/2$**

- $x = (1 \pm \sqrt{(1 + 4)})/2$

3. **54**

- $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 = 54$

4. **$1/\phi = (\sqrt{5} - 1)/2 = \phi - 1$**

- $1/\phi = 2/(1 + \sqrt{5}) = 2(\sqrt{5} - 1)/((\sqrt{5} + 1)(\sqrt{5} - 1)) = 2(\sqrt{5} - 1)/4 = (\sqrt{5} - 1)/2$.
- $\phi - 1 = (1 + \sqrt{5})/2 - 1 = (\sqrt{5} - 1)/2$. They are equal.

Task A: The ratio limit

a. **1, 2, 1.5, 1.667, 1.6, 1.625**

- $1/1, 2/1, 3/2, 5/3, 8/5, 13/8$

b. **$r_7 = 21/13$**

- $F_{n+2}/F_{n+1} = (F_{n+1} + F_n)/F_{n+1} = 1 + 1/r_n$.
- $r_7 = 1 + 8/13 = 21/13$

c. **$L = (1 + \sqrt{5})/2 = 1.618$**

- In the limit $L = 1 + 1/L$, so $L^2 = L + 1$.
- $L = (1 \pm \sqrt{5})/2$. Every r_n is positive, so L cannot be negative: $L = (1 + \sqrt{5})/2 = 1.618$ (3 d.p.).

d. **$r_{10} = 89/55$; about 0.00015 above L**

- $F_{10} = 55, F_{11} = 89$, so $r_{10} = 89/55 = 1.61818...$
- $1.61818... - 1.61803... = 0.000148...$, which is 0.00015 (2 s.f.).

Task B: Periodic and patterned sequences

a. **-1, 1/2, 2: period 3**

- $u_2 = 1/(1 - 2) = -1$
- $u_3 = 1/(1 + 1) = 1/2$
- $u_4 = 1/(1 - 1/2) = 2 = u_1$, so the sequence repeats every 3 terms.

b. **15**

- Each block of three terms adds to $2 - 1 + 1/2 = 3/2$.
- 30 terms make 10 blocks: $10 \times 3/2 = 15$

c. **67**

- F_n is even exactly when 3 divides n : 33 of the first 100.
- $100 - 33 = 67$ are odd.

Task C: The golden ratio in a pentagon

a. **108°**

- The interior angles add to $(5 - 2) \times 180^\circ = 540^\circ$, so each is $540^\circ \div 5 = 108^\circ$.

b. **1.618**

- $d^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos 108^\circ = 2 - 2 \cos 108^\circ = 2.618\dots$
- $d = 1.618$ (4 s.f.): the golden ratio.

c. **$d = 2 \cos 36^\circ = (1 + \sqrt{5})/2 = \phi$**

- The diagonal is the base of an isosceles triangle with two sides 1 and apex angle 108° , so its base angles are 36° .
- Dropping a perpendicular from the apex splits the base into two halves of length $\cos 36^\circ$, so $d = 2 \cos 36^\circ$.
- $2 \times (1 + \sqrt{5})/4 = (1 + \sqrt{5})/2 = \phi$

Extension

1. **$A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$**

- $\phi^{n+2} = \phi^n \phi^2 = \phi^n(\phi + 1) = \phi^{n+1} + \phi^n$, and the same for ψ , so any combination works.
- $A\phi + B\psi = 1$ and $A\phi^2 + B\psi^2 = 1$. Subtracting gives $A + B = 0$ (because $\phi^2 - \phi = \psi^2 - \psi = 1$).
- So $A(\phi - \psi) = 1$, and $\phi - \psi = \sqrt{5}$: $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$.

2. **$F_{25} = 75\,025$**

- $|\psi^n/\sqrt{5}| < 1/\sqrt{5} < 1/2$ for every $n \geq 1$, so F_n is within $1/2$ of $\phi^n/\sqrt{5}$.
- $\phi^{25}/\sqrt{5} = 75\,024.999997\dots$, so $F_{25} = 75\,025$.

Every question is original and every answer was recomputed by computer before publishing.